Developing the Concept of Place Value

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What a study of the historical development of mathematical concepts can offer teaching is still being debated. This study examines use of a combination of the historical development of number systems and modelling, with concrete materials as a way of deepening students' understanding of positional notation. It looks at place value in different number bases as a way of enhancing students' understanding of the decimal number system. The results suggest that the combination of a historical and a concrete approach helped the students to understand the place value system to the extent that they could generalise it to other bases.

Background

The understanding of the concept of a positional, or place value, system is central to developing number sense and is also the basis for the four fundamental operations on numbers, as confirmed by the concept map research of Schmittau and Vagliardo (2006, p. 7), who have shown "the centrality of positional system in the conceptually dense system of concepts that comprise elementary school mathematics. Not only does it connect to many important concepts … it is also a prerequisite for any real understanding of the base ten system". However, anecdotal and other evidence (Thomas, 2004) suggest that this vital and central concept is not well understood by students. One reason is that the concept of positional system cannot be developed through the teaching of base ten alone, and students cannot completely understand the decimal system unless it is seen as a particular case of a more general concept of positional notation. Thus this stresses the need for teaching of multiple bases to help students develop the concept of positional system. In addition, since a positional system is a superordinate concept, founded on multiple basic concepts, in order to understand it one must have rich foundational schemas. Unfortunately, one cannot just define such a concept into existence for students since "concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples" (Skemp, 1971, p. 32).

Not only is the knowledge of multiple bases vital for understanding the concept of place value, but it also serves as a foundation for the development of other crucial concepts, such as variable, exponent, polynomial, and polynomial operations, amongst others. Students' difficulties in algebra and these areas have been well documented (e.g., Kieran, 1992; MacGregor & Stacey, 1994; Warren, 2003) and educators' views on the various approaches to beginning algebra, such as generalisation, problem solving, and function/modelling, are also clear in the literature (e.g., Mason, 1996; Radford, 1996; Ursini, 2001). According to Mason (1996), generalisation is the heartbeat of mathematics and that "expressing generality is central to all mathematics, including arithmetic" (Mason, Graham, & Johnston-Wilder, 2005, p. 95). He goes on to state that one of the most important sources of generalization is the domain of number and, in detecting and expressing number patterns, general number can be seen as a pre-cursor of variable, the central concept of algebra. Hence a good knowledge of positional notation could assist in a smoother transition to algebra through a consideration of multiple bases to the notion of a general number base *n*.

In this research study we considered the importance of understanding of positional notation and how it might be improved using a combination of concrete materials, multiple representations and historical perspectives. The first of these has been appreciated since the time of Piaget's description of the concrete operational stage of learning, because for these students one must recognise that "in sum, concrete thought remains essentially attached to empirical reality" (Inhelder & Piaget, 1958, p. 250). At one time materials such as Dienes blocks (Dienes, 1960) were widely used but have since grown unfashionable. Secondly, representational versatility (Thomas, 2006) lies at the heart of much of what mathematics is. Students may interact with a representation by observing it, for example by noticing properties of the representation itself or of the conceptual processes or object(s) represented, or acting on it. The versatility arises in the ability to translate between representations of the same concept and to interact with these representations in qualitatively different ways. The third aspect is the use of history to inform practice. In recent years, there has been a continuing tradition of using history of mathematics in the teaching and learning of mathematics. Educators and researchers (Fauvel & van Maanen, 2000; Gupta, 1995; Katz, 2001) have asserted that the history of mathematics is an excellent resource for motivating students to learn mathematics, and one of the greatest benefits is in enhancing the understanding of mathematics itself. Of course there are different ways in which historical material may be incorporated in the classroom, with the history implicit or explicit in the teaching situation (Fauvel & van Maanen, 2000). Either way it can bring about a global change in the teacher's approach. This is because a historical and epistemological analysis (Puig & Rojano, 2004) may help the teacher to understand stages in learning (Barbin, 2000) and why a certain concept is difficult for the student. In turn this can help with teaching strategy and development. A specific example of the implicit use of history is the historical development of the present day decimal number system.

A review of some texts (Datta & Singh, 2001; Joseph, 2000; Srinivasaiengar, 1967) reveals that the decimal number system with place value and zero used today originated in India, and this system was passed on to the Arab mathematicians who then carried the system to Europe. A study of this history reveals that the "perfection" of the number system was preceded by centuries of experience of working with very large numbers (as part of solving problems in astronomy). The ancient Indian mathematicians developed a scientific vocabulary of number names including names for powers of 10, even going up to 10^{53} and this consideration of large numbers and exponential multiplication and its symbolisation seems to have prompted the creation of zero and the number system with place value (Datta & Singh, 2001). Although the *rhetorical, syncopated,* and *symbolic* stages are usually associated with algebra (Kieran, 1992), they seem to have also been present in the realm of number in Indian history of mathematics. In addition, studying different number systems from history provides students with the opportunity of developing an understanding of the concept of numerals as number symbols, as well as the principles that were used with these symbols. Moreover, the study of number systems from history presents mathematics as a human endeavour with twists and turns, false paths, and dead-ends, and helps learners towards a more realistic appreciation of their own attempts.

In some countries, including New Zealand, the teaching of multiple bases is no longer present in most mathematics textbooks at the Primary and Intermediate level and so is not taught in schools. Hence this research study sought to use concrete materials, the theory of representations, and both explicit and implicit historical analysis, in the classroom for the concept of place value. We addressed the question of whether such an approach could help to improve students' understanding of this positional system of representing number.

Method

The research study comprised a case study of a class of 27 Year 9 (age 13 years) students at a decile 5 (middle socio-economic level) at a secondary school in Auckland, New Zealand. This class, called the "Global" class, was a new concept in 2005, with students from many different cultures and ethnicities, and a "global" approach to core subjects. The class used in the research thus represents a wide variety of cultural backgrounds, including Indonesian, Russian, Hungarian, Dutch, American, Malaysian, Zimbabwean, Chinese, Korean, Japanese, Cypriot, Swedish, Maori, Pacific Island, and New Zealand European students. However, most of the students had their intermediate schooling in New Zealand and hence were proficient in English. The exceptions to this were two Korean students and a Chinese student who had only recently arrived in the country, who were taking ESOL classes. Possibly due to a positive attitude the class was performing above average for the year group in the school. The teacher explained to the students what was going to be taught and why it was important to their learning. The classroom process was very much task oriented, and all the class lessons were taught by the first-named researcher. The first task was intended to get students to think about the need for a number system and how it might have been constructed. To accomplish this they were encouraged to work in groups of 2, 3, or 4 and try to create a number system of their own. This included deciding on the grouping size, the number of symbols needed, and how they would represent and add numbers. The students were given a large number of coloured sticks to help with their thinking and sheets of paper on which to write their ideas. Following this the students were given a pre-test comprising questions that addressed their current understanding of place value. A sample of the kind of questions used is given in Figure 1.

Following the test the students' second task was to investigate the number systems of past civilisations to see what could be learned from them. Having considered the numbers 0-10 in their own languages, including writing down the number symbols in their language on the board, and saying the numbers, they then spent five to six lessons of 60 minutes each working through worksheets on different number systems from around the world and from different time periods. These included Primitive, Egyptian, Babylonian, Roman, Greek, and Mayan, and finally the present Indian decimal system. The tasks involved them writing numbers in the different systems, only two of which had a place value (Babylonian and Mayan). Following the investigation of each number system the students discussed the symbols in the system, along with general features such as place value and zero, its advantages and disadvantages, and then wrote down their observations on the system.

The third task, comprising two lessons, was to use concrete materials to analyse base 10 numbers. The students were given large numbers of coloured sticks and were asked to group the sticks in tens and then hundreds, thousands (they managed one ten thousand!) etc., tying the sticks with elastic bands that they were given (parts of sticks were used for tenths and hundredths) and they used them to model numbers, such as 12386. Keeping in mind that the historical development through the rhetorical stage was in place for a long time, the sticks were used to model numbers in the same way that we say the number, that is, one set of ten thousand and two sets of one thousand (1 set of $10\times10\times10\times10$ and 2 sets of $10\times10\times10$).

Section A

1. Write the following in words.
   a) 7905 ______
   b) 100005 ______
2. Write the following in numerals
   a) fifty three thousand eight hundred and ninety two ______
   b) sixty two thousand and nine ______
3. For the following numbers, what is the actual value for each of the digits?
   a) 35275 b) 6008 c) 7658.32
4. What is the meaning of zero in question 3b?
5. How many symbols do we have in the number system that we use? ______
6. What is the base of the number system that we use? ______
7. How many symbols do we need for a number system with base six? ______
8. How many symbols do we need for a number system with base forty three? ______

Section B

1. Suppose we consider a number system with base **six**. Write the following numbers in words.
   a) 3524 ______
   b) 40035 ______
   c) 324.15 ______
2. Suppose we consider a number system with base **seven** where

   1 is written as 1
   2 is written as Γ
   3 is written as Δ
   4 is written as □
   5 is written as ∃
   6 is written as ϖ
   0 is written as 0

   Then write the following numbers in words.

   a) □ϖΔΓ b) Δ0□ϖ c) ΓΔ0.∃□

Figure 1. Some of the pre- and post-test questions.

In the next stage of this task, only a single bundle of sticks was placed to represent the place value. For example, only one bundle of 10 was placed and 8 sticks were placed underneath it to represent 80. During the final stage of the task the bundle of 10 was removed and students had to imagine the value of the place. Examples of two of these representations of the number 234.23 are given in Figure 2 (the decimal point is represented by a band). The cognitive linking of these representations is a key step in the construction of the system. There was also discussion surrounding the need for a symbol for zero when we consider a number such as 407.

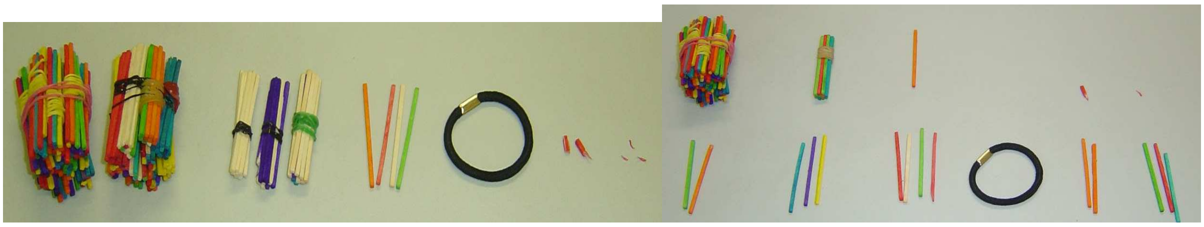
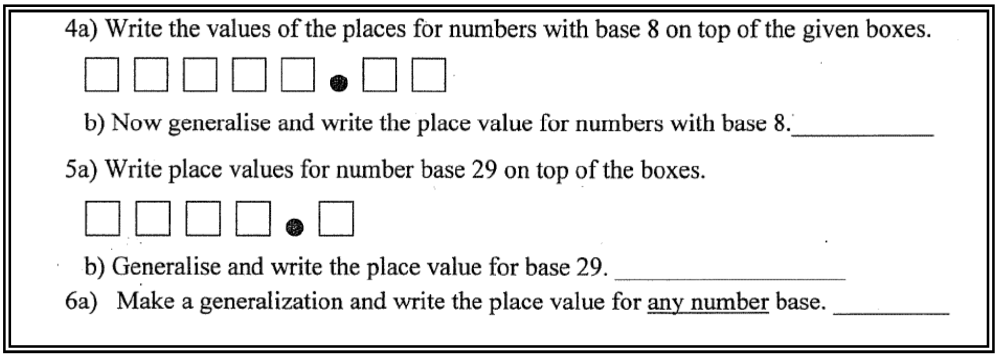


Figure 2. Two different representations of 234.23 used in task 3.

Following these tasks the students were given a post-test, along with extra questions on generalisation (see Figure 3 for some of these questions) involving bases 6, 7, 8, and 29 as well as base 10, and also asking for a generalisation.



4a) Write the values of the places for numbers with base 8 on top of the given boxes.

b) Now generalise and write the place value for numbers with base 8.____________

5a) Write place values for number base 29 on top of the boxes.

b) Generalise and write the place value for base 29. ____________________

6a) Make a generalization and write the place value for <u>any number</u> base. __________

Figure 3. Some of the “extra” post-test questions.

Due to the difficulty of the extra questions some students requested further explanation on the idea of generalization, so the teacher used half a lesson to put up some patterns on the board that the students had to generalise. She explained that she wanted them to look at the patterns, say what they saw and then write a sentence with symbols that would represent any one or all of the lines. They discussed what “make a generalisation”, “in general” and “generalise” means. The patterns below were put up on the board and students had to verbalise as to what was the same and what was changing across any line and generalise. Then they had to look at the vertical line on the right and generalise further for any base *a*. Finally, the students were allowed to answer the extra questions one more time.

$$\begin{array}{lllllllll}
7^4 & 7^3 & 7^2 & 7^1 & 7^0\,. & 7^{-1} & \ldots\ldots\ldots & & 7^k \\
5^4 & 5^3 & 5^2 & 5^1 & 5^0\,. & 5^{-1} & \ldots\ldots\ldots & & 5^k \\
34^4 & 34^3 & 34^2 & 34^1 & 34^0. & 34^{-1} & \ldots\ldots\ldots & & 34^k \\
 & & & & & & & & a^k
\end{array}$$

Results

In order to establish some comparative baseline data on Year 8 students' understanding of place value we accessed the results of the Assessment Tools for Teaching and Learning (asTTle) (Hattie, Brown, & Keegan, 2005) standardised tests for the whole Year 9 group at the school the Global class attended. Five of these questions were on the topic of place value and hence exposed areas of difficulty for students of this age and background, forming a comparative population. The students all sat the tests on the first day of their school year, before the research study took place. Results on questions 8 and 22 were combined on the test, since both address the skill "Explain the meaning of digits in numbers up to 3 decimal places", and could not be separated. Question 8 essentially asked whether 1.35 or 1.342 is larger, and question 22 asked students to write a number with 1 in the hundredths column, 2 in the tens, 5 in the thousandths, 6 in the ones and 9 in the tenths. Similarly questions 11 and 23 considered "Order decimals up to 3 decimal places" and question 13, "Explain the meaning of the digits in any whole number". Table 1 shows the comparison of the Global class results with those of the rest of the year group. These show that on questions 8 and 22 (χ^2=9.95, p<0.01) the Global class performed significantly better than the year group. However, on question 13 (χ^2=0.45, ns) and questions 11 and 23 (χ^2=3.06, ns), there was no significant difference in performance. Two comments may be made on this. Firstly it confirmed the view that the Global class was performing a little above average for their year group, and secondly that these place value skills are a problem for many students of this age.

Table 1

A Comparison of Year 9 Students with Global Class on Place Value

Question	Year 9 Group (N=125) % Wrong	Year 9 Global Class (N=27) % Wrong
8 and 22	72%	54%
13	72%	66%
11 and 23	51%	29%

Work on the Tasks

When we look at what the students produced for their number systems on the first task, most simply took the base 10 system and created their own symbols (Figure 4, row 1). Others (Figure 4, row 2) employed an additive system using a symbol for ten as their base to get 39. The only group who tried to do anything differently is shown in Figure 4 row 3. They used a system of merging two symbols together into a partial multiplicative arrangement, but they still have a new symbol for 36 and are not using place value. However, this was the first task that the students worked on and it accomplished its purpose of getting them to think about number systems and how they are constructed.

The second task on considering how the different number systems developed historically proved interesting to the students, for differing reasons. Some liked particular symbols such as the Egyptian and the Roman for aesthetic reasons, and others felt that some systems, such as Roman and Primitive systems, were easier to use, whereas others found the Mayan system difficult and confusing. However, when asked to represent large numbers students realised they had to repeat symbols many times and also had to create more and more symbols (see the sample comments of S1 and S24 in Figure 5). When asked why they were able to write large numbers with only ten symbols in the present decimal system, students found the question quite challenging and one student said "it was because of all the zeros".

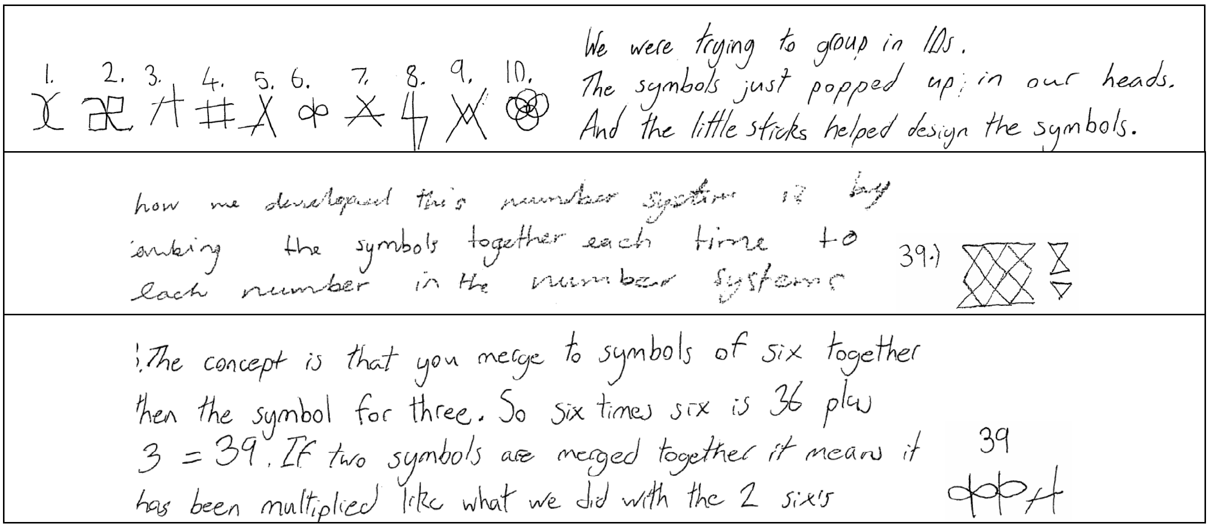


Figure 4. Students' work on creating their own number system.

During episodes of teacher intervention during the work with the groups of coloured sticks, different numbers were modelled on the board in base 10 (for example 10^3 was also written as $10\times10\times10$, $10^2\times10$, 1000, and in words), leading to a discussion of exponential multiplication and place value. This was done so that students not only see one thousand as a thousand ones, but also as 10 groups of 10 groups of 10. The following was written up on the board for each one of the positions.

Thousand 1000 $10^2\times10$ $10\times10\times10$	10^4 $\boxed{1}$	10^3 $\boxed{4}$	10^2 $\boxed{8}$	10^1 $\boxed{6}$	10^0 $\boxed{7}$ •	10^{-1} $\boxed{2}$

S24 Babylonian	Sort of ok, sort of hard. They had place value but not a real zero. Grouping in large numbers is hard. This sign was used as a separater symbol.
Roman S24	8. Difficult to write large numbers but they didn't large numbers
S1 Babelonean	7. Even though only two symbols were used, we had to write a lot of ~~numbers~~ symbols to express large number.
S1 Hindu-Arabic	3. We didn't have to keep making up new symbols – because the system has place value and zero.

Figure 5. Two students' observations on historical number systems.

The tasks gave students an opportunity to construct other concepts, such as the relative sizes of numbers like 10^4 and 10^{-2}. That they were engaging with these ideas was shown by comments about the "bigness" of something like 10^{12} and 10^{53} and the smallness of 10^{-23}! In the second session of this task exactly the same procedure was followed, but this time the students grouped the sticks in sets of 6s, 36s, 216s, etc., and hence different numbers were represented in base six. Again this was written on the board as above in different representations: in words, exponential forms, and full forms, (e.g., 4 lots of 216 (6^3), 5 lots of 36 (6^2)). There was discussion on the word *base* and how many symbols were needed for a particular base. When working with the groups of coloured sticks and by looking at the patterns, students came up with 10^0 as 1 and one tenth as 10^{-1}. It was brought to the attention of students that in a number such as 12796.34 the three sticks used represented 3 lots of the tiny bits of sticks, or 10^{-1}.

Some of the students commented that they found the work on the tasks, and especially the "project" to create their own number system, enjoyable and fun, stating: "This is lots of fun. Got us thinking about funny names and symbols" (S3); "This is fun. We like working together and bounce ideas off each other but it is hard. It is like making your own language up" (S5); "It was fun. Kind of interesting figuring out what symbols to use. A great way to get creative" (S23); and "Very interesting. Sticks helped us to think. I felt I was designing something for the future" (S24). As S5 observed, it was also challenging for them, to the point that some found it very difficult and others felt out of their depth. This was occasionally, according to S9 and S12 because their group did not work so well together: "Extremely hard to create own number system. The group were not communicating very well as all of us were thinking differently and it was hard to co-ordinate our ideas and write them down"(S12); and "The group was confused. Different opinions in the group and they all wanted different things/symbols"(S9).

Others also said how they found the work "challenging" (S13) or "quite hard" (S21), or they were "Confused. Concerned I was not doing anything" (S27). Student S21 was the only one who was negative throughout the whole unit of work and it was very difficult to help her. She felt she was not good at mathematics and she said she did not care about mathematics anyway. In summary, we can say that the task was stimulating but not easy for this group of students.

Test Results

From the pre-test to the post-test all students except for S13 improved their scores, and overall there was a significant improvement in the mean score on the test ($Mean_{pre}$=7.41, $Mean_{post}$=13.63, t=6.22, p<0.0001). There was improvement on every question on the tests (sections A and B), but especially on section A, questions 7 and 8 (from 5 and 3 correct to 23 and 20, respectively), and every question in section B (from 0 on every question to scores from 15 to 17 correct). Questions 7 and 8 asked how many symbols are need for bases 6 and 43, and this generalisation was clearly better understood after the module of work. Two students, S6 and S19, are attending ESOL classes and were very hindered by language difficulties. Although they only attempted to answer some of the questions they did both improve, from 0 each on the pre-test to 6 and 7 respectively on the post-test. It was pleasing to see that by the end of the module of work 23 of the students could answer Q4a) for base 8 and 19 of these could generalise the place value to 8^x (Q4b)), or equivalent. Similarly 24 students could do the same for base 29 (Q5a)), 21 of these could generalise the place value here too to 29^x, and the same number could even take this to any base and write n^x (Figure 6).

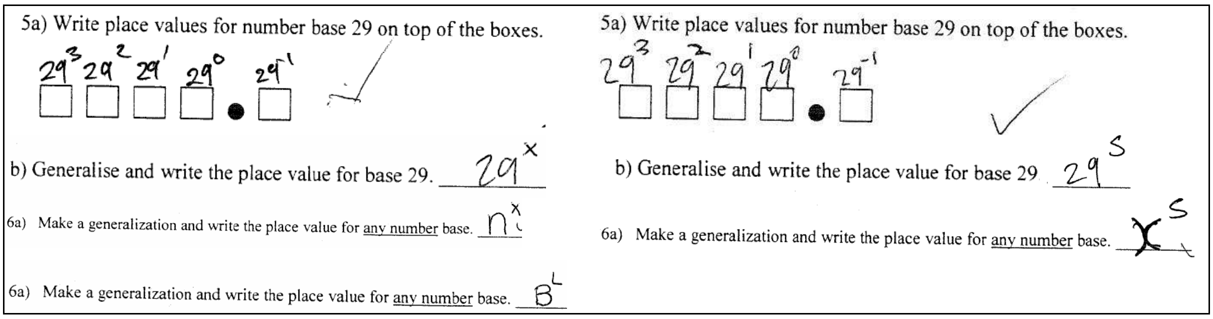


Figure 6. The generalisations of two students.

A number of students, S2, S3, S5, S9, S16, S17, S26, and S27, all expressed the thought that they had found the use of the sticks helpful to formulate their thinking, commenting that "I think the sticks helped me learn about doing place value in different bases" (S2), "The sticks helped me visualise the challenge" (S3), "With the sticks it was easier because we saw what we were doing not just hearing it" (S5), "When you do it with the sticks it helped because you learn better when you do stuff in person, using your hands" (S16). Only a couple of students (S22 and S25) mentioned negative aspects of the sticks, saying how "the sticks didn't help me much" (S22) or how they found the sticks "confusing" (S25). Some also mentioned that they had enjoyed and benefited from the historical ideas they had engaged with: "the different systems were quite fun because we now know how some other cultures write and do their systems" (S5); "the different number systems have made me realise how [much] easier our number system is" (S11); "I learnt how much they struggled to accomplish these historic number systems" (S12); and "Using the other number systems was fun" (S20).

Conclusions

We suggest that the importance of the understanding of place value cannot be underestimated, as Schmittau and Vagliardo's (2006) research on concept mapping confirms. This study attempted to develop in students a meaningful understanding of place value and a structure of the number system through: considerations of large numbers and exponential multiplication; use of concrete materials, multiple bases, multiple representations; and a review of development of historical number systems. The focus was on students' understanding of structure and recognition that the numerals that they deal with on a daily basis are number symbols forming part of a system. The results show that students achieved a certain measure of success and were able to generalise the multiplicative (including exponential) structure of the number system. The study also shows that students respond well when extended beyond what they are responsible for in terms of learning in order to conceptualise what they *have* to learn in the curriculum. This may have implications for mathematics curriculum development, as the positional system receives superficial treatment from most mathematical textbooks. The research suggests that if students are to develop meaning for place value then the topic should be included in the curriculum, since a failure to develop understanding of positional notation adequately will restrict future learning in mathematics.

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